Advanced techniques to compute improper integrals using a CAS

Gabriel Aguilera, José Luis Galán, María Ángeles Galán,
Yolanda Padilla, Pedro Rodríguez
University of Málaga (Spain)

Ricardo Rodríguez
Technical University of Madrid

jl_galan@uma.es

Abstract

Let us consider the following types of improper integrals:

$$\int_0^\infty f(t)\,\mathrm{d}t \quad ; \quad \int_{-\infty}^0 f(t)\,\mathrm{d}t \quad \text{and} \quad \int_{-\infty}^\infty f(t)\,\mathrm{d}t$$

Let F be an antiderivative of f. The basic approach to compute such integrals involves the following computations:

$$\begin{aligned}
\int_0^\infty f(t)\,\mathrm{d}t &= \lim_{m\to\infty}\int_0^m f(t)\,\mathrm{d}t = \lim_{m\to\infty}\Big(F(m)-F(0)\Big)\\
\int_{-\infty}^0 f(t)\,\mathrm{d}t &= \lim_{m\to-\infty}\int_m^0 f(t)\,\mathrm{d}t = \lim_{m\to-\infty}\Big(F(0)-F(m)\Big)\\
\int_{-\infty}^\infty f(t)\,\mathrm{d}t &= \int_{-\infty}^0 f(t)\,\mathrm{d}t + \int_0^\infty f(t)\,\mathrm{d}t \qquad \text{or, in case of convergence,}\\
\int_{-\infty}^\infty f(t)\,\mathrm{d}t &= \lim_{m\to\infty}\int_{-m}^m f(t)\,\mathrm{d}t = \lim_{m\to\infty}\Big(F(m)-F(-m)\Big) \qquad \text{(Cauchy principal value)}
\end{aligned}$$

But, what happens if an antiderivative F for f or the above limits do not exist?

For example, for $\displaystyle\int_0^\infty \frac{\sin(at)}{t}\mathrm{d}t$; $\displaystyle\int_0^\infty \frac{\cos(at)-\cos(bt)}{t}\mathrm{d}t$ or $\displaystyle\int_{-\infty}^\infty \frac{\cos(at)}{t^2+1}\mathrm{d}t$ the antiderivatives can not be computed. Hence, the above procedures cannot be used for these examples.

In this work we will deal with advance techniques to compute this kind of improper integrals using a CAS. Laplace and Fourier transforms or Residue Theorem in Complex Analysis are some advance techniques which can be used for this matter.

We will introduce the file `ImproperIntegrals.mth`, developed in DERIVE 6, which deals with such computations.

Some CAS use different rules for computing integrations. For example RUBI system, a **ru**le-**b**ased **i**ntegrator developed by Albert Rich (see `http://www.apmaths.uwo.ca/~arich/`), is a very powerful system for computing integrals using rules. We will be able to develop new rules schemes for some improper integrals using `ImproperIntegrals.mth`. These new rules can extend the types of improper integrals that a CAS can compute.

Keywords

Improper integrals, DERIVE, CAS, Laplace Transform, Fourier Transform, Residue Theorem